Bending of a Uniformly Loaded Circular Plate with Mixed Boundary Conditions

BERNHARD STAHL*
Amoco Production Company, Tulsa, Okla.
AND
LEON M. KEER†
Northwestern University, Evanston, Ill.

This paper treats the bending of a uniformly loaded circular plate with mixed conditions on the boundary. Two cases are considered: 1) clamped—simply supported and 2) simply supported—free. Deflections and bending moments are calculated at the center of the plate and compared to results obtained by other investigators. Excellent agreement is found except when the clamped segment or support segment is short compared with the circumference. An auxiliary function, which permits the stress intensity factor to be determined, is also tabulated for each case.

Nomenclature

a = radius of plate
D = flexural rigidity of plate
$E(\ )$ = complete elliptic integral of the second kind
$F(\ )$ = complete elliptic integral of the first kind
h = plate thickness
$H(\ )$ = Heaviside function
I_i = modified Bessel function of the first kind and ith order
J_i = Bessel function of the first kind and ith order
L_1 = modified Struve function of the first order
$M_{rr}, M_{\theta\theta}$ = moments per unit length of plate
q = uniform load on circular plate
r = radial coordinate of plate; also dummy variable in integral equation
t = dummy variable, $t = \alpha\rho$
V_r = Kirchhoff shear per unit length of plate
δ = coordinate measured from and perpendicular to middle plane
θ = circular coordinate of plate
ν = Poisson's ratio, assumed to be 0.3
ρ = dummy variable
$\phi(t), \psi(\rho)$ = auxiliary functions, $\phi(\alpha\rho) = \psi(\rho)$
∇^4 = biharmonic operator

Introduction

A NUMBER of investigators have previously treated the bending of a circular plate with mixed boundary conditions. Nowacki and Olesiak[1] have considered the vibration, buckling and bending of a circular plate. More recently, Conway and Farnham[2] and Leissa and Clausen[3] have given numerical solutions to the static bending of a circular plate with mixed conditions. One method used by these authors allowed the problem to be reduced to finding the solution to a Fredholm integral equation of the first kind from which an unknown, e.g., the moment distribution along a clamped segment, could be obtained directly. Another method used was the point matching technique wherein the boundary conditions are satisfied at prescribed points in a direct manner or least squares sense. The present analysis differs from the preceding in that the problem is reduced to a Fredholm integral equation of the second kind for an unknown auxiliary function. The method requires that the singular part of the solution be isolated and treated analytically, thereby permitting information such as the stress intensity factor to be obtained.

Bartlett[4] has considered problems of vibration and buckling of a circular plate clamped on part of its boundary and simply supported on the remainder. Dual series equations were used for the problem formulation and a variational method was used to approximate the lowest eigenvalue. Subsequently Noble[5] provided an alternative solution by use of a closed-form representation for a trigonometric series. Although the present paper only treats the static problem, the method of solution can be extended to include eigenvalue problems also.

Two cases of boundary conditions are considered in this paper: 1) clamped—simply supported and 2) simply supported—free. The boundary conditions are arranged so that the bending is symmetric about two mutually perpendicular axes. Numerical results are provided for both cases and are compared with the results of other investigators.

The present work is restricted to classical plate theory with the notation of Timoshenko and Woinowsky-Krieger.[6] The bending of the plate is governed by

$$D\nabla^4 w = q \tag{1}$$

and the general solution which exhibits the two-fold symmetry of the problem is

$$w = \sum_{m=0,2,4,\ldots}^{\infty} (A_m r^m + B_m r^{m+2})\cos m\theta + \frac{qr^4}{64D} \tag{2}$$

The unknown coefficients A_m and B_m are determined by satisfying the boundary conditions on the circular edge.

Certain integral representations of series involving Bessel functions will be required for the subsequent analyses. They are listed here for convenience and will be referenced as needed:

$$\sum_{m=2,4,\ldots}^{\infty} J_0(mt)\cos(m\theta) = 0.5[(t^2-\theta^2)^{-1/2}H(t-\theta)-1] \qquad \theta+t<\pi \tag{3}$$

$$\sum_{m=2,4,\ldots}^{\infty} J_0(mt)\sin(m\theta) = 0.5(\theta^2-t^2)^{-1/2}H(\theta-t) - \int_0^{\infty}[\exp(\pi s)-1]^{-1}I_0(ts)\sinh(\theta s)\,ds, \qquad \theta+t<\pi \tag{4}$$

Received August 9, 1971; revision received January 7, 1972.
Index category: Structural Static Analysis.
* Senior Research Engineer. Member AIAA.
† Professor, Department of Civil Engineering.

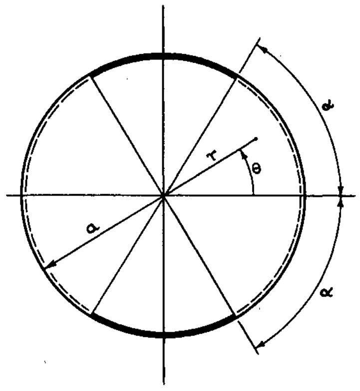

Fig. 1 Clamped—simply supported (simply supported—free) circular plate.

$$\sum_{m=2,4,\ldots}^{\infty} J_1(mt)\sin(m\theta) = 0.5\theta t^{-1}(t^2-\theta^2)^{-1/2}H(t-\theta), \qquad \theta+t<\pi \tag{5}$$

$$\sum_{m=2,4,\ldots}^{\infty} J_1(mt)\cos(m\theta) = 0.5t^{-1}-0.5\theta t^{-1}(\theta^2-t^2)^{-1/2}H(\theta-t) - \int_0^{\infty}[\exp(\pi s)-1]^{-1}I_1(ts)\sinh(\theta s)\,ds, \qquad \theta+t<\pi \tag{6}$$

$$\sum_{m=2,4,\ldots}^{\infty} m^{-1}J_1(mt)(-1)^{m/2} = -t/4, \qquad t<\pi/2 \tag{7}$$

These identities can be derived by contour integrations of appropriate integrands around the first quadrant of the complex plane. For example, Eq. (3) may be derived by a contour integration of

$$\frac{\exp[i(\pi/2)z]J_0(tz)\cos(\theta z)}{\sin[(\pi/2)z]}$$

and by use of the identity[7]

$$\int_0^{\infty} J_0(tu)\cos(\theta u)\,du = (t^2-\theta^2)^{-1/2}H(t-\theta)$$

Clamped—Simply Supported Circular Plate

Because of the two-fold symmetry of the problem, it is only necessary to consider one quadrant of the circular plate. The boundary conditions are illustrated in Fig. 1 and are given as follows on $r=a$:

$$w=0, \qquad 0\leqq\theta\leqq\pi/2 \tag{8}$$

$$\partial w/\partial r=0, \qquad \alpha<\theta\leqq\pi/2 \tag{9}$$

$$M_{rr} = -D\{\partial^2 w/\partial r^2+\nu[(1/r)\,\partial w/\partial r+(1/r^2)\,\partial^2 w/\partial\theta^2]\}=0, \qquad 0\leqq\theta<\alpha \tag{10}$$

Boundary condition (8) leads to the relations

$$A_0 = -B_0a^2-qa^4/64D, \qquad A_m=-a^2B_m \tag{11}$$

The remaining boundary conditions lead to the governing dual series equations

$$\sum_{m=0,2,4,\ldots}^{\infty} P_m\cos m\theta = 0, \qquad \alpha<\theta\leqq\frac{\pi}{2} \tag{12}$$

$$\sum_{m=0,2,4,\ldots}^{\infty} m\left[1+\frac{(1+\nu)}{2m}\right]P_m\cos m\theta = -\frac{1}{16}, \qquad 0\leqq\theta<\alpha \tag{13}$$

where

$$B_0 = 0.5qa^2(P_0-\tfrac{1}{16})/D, \qquad B_m = 0.5a^{2-m}qP_m/D \tag{14}$$

The solution to the preceding dual series equations begins by choosing

$$P_0=\frac{1}{2}\int_0^{\alpha}\phi(t)\,dt, \qquad P_m=\int_0^{\alpha}\phi(t)J_0(mt)\,dt \tag{15}$$

which satisfy Eq. (12) by virtue of Eq. (3). This choice also provides a square root moment singularity at the tips of the clamped segments, in agreement with the infinite plate solutions.[8-10] Rewriting Eq. (13) in the form

$$\frac{d}{d\theta}\sum_{m=0,2,4,\ldots}^{\infty} P_m\sin m\theta+\frac{(1+\nu)}{2}\sum_{m=0,2,4,\ldots}^{\infty} P_m\cos m\theta = -\frac{1}{16}, \qquad 0\leqq\theta<\alpha \tag{16}$$

substituting Eqs. (15, 3, and 4), and integrating once between 0 and θ, yields

$$\int_0^{\theta}(\theta^2-t^2)^{-1/2}\phi(t)\,dt = -\frac{(1+\nu)}{2}\int_0^{\theta}\int_s^{\alpha}(t^2-s^2)^{-1/2}\phi(t)\,dt\,ds - \frac{\theta}{8}+2\int_0^{\alpha}\phi(t)\int_0^{\infty}[\exp(\pi s)-1]^{-1}I_0(ts)\sinh(\theta s)\,ds\,dt \tag{17}$$

Equation (17) is in the form of Abel's integral equation

$$\int_0^{\theta}(\theta^2-t^2)^{-1/2}\phi(t)\,dt = h(\theta) \tag{18}$$

which has the solution

$$\phi(t)=\frac{2}{\pi}\frac{d}{dt}\int_0^{t}(t^2-\theta^2)^{-1/2}\theta h(\theta)\,d\theta \tag{19}$$

After some manipulations and with the help of certain identities,[11,12] the final result becomes a Fredholm integral equation of the second kind

$$\psi(\rho)+\int_0^1 K(\rho,r)\psi(r)\,dr = f(\rho) \qquad 0\leqq\rho\leqq 1 \tag{20}$$

where

$$K(\rho,r)=\left(\frac{1+\nu}{\pi}\right)\alpha\begin{bmatrix}\dfrac{\rho}{r}F\left(\dfrac{\rho}{r}\right), & r>\rho\\ F\left(\dfrac{r}{\rho}\right), & r<\rho\end{bmatrix} - 2\alpha^2\rho\int_0^{\infty}[\exp(\pi s)-1]^{-1}I_0(\alpha rs)sI_0(\alpha\rho s)\,ds \tag{21}$$

and

$$f(\rho) = -\alpha\rho/8 \tag{22}$$

To obtain deflections and bending moments, it is necessary to solve Eq. (20) numerically. At the center of the plate these physical quantities are particularly easy to calculate since $r=0$ and thus only the first one or two terms of the infinite series contribute to the final result, as follows:

$$w|_{r=0} = -B_0a^2-a^4q/64D \tag{23}$$

$$M_{rr}\big|_{\substack{r=0\\ \theta=0}} = -2D[(1+\nu)B_0+(\nu-1)a^2B_2] \tag{24}$$

and

$$M_{\theta\theta}\big|_{\substack{r=0\\ \theta=0}} = -2D[(1+\nu)B_0+(1-\nu)a^2B_2] \tag{25}$$

The singularity of the moment field M_{rr} along $r=a$ may be illustrated by use of the expression

$$M_{rr}|_{r=a} = -qa^2\left\{(1+\nu)P_0+\frac{1}{8}+2\sum_{m=2,4,\ldots}^{\infty} mP_m\cos m\theta+(1+\nu)\sum_{m=2,4,\ldots}^{\infty}P_m\cos m\theta\right\} \tag{26}$$

Integrating the second expression in Eq. (15) by parts leads to

$$P_m=\frac{\phi(\alpha)J_1(m\alpha)}{m}-\frac{1}{m}\int_0^{\alpha}tJ_1(mt)\frac{d}{dt}\left(\frac{\phi(t)}{t}\right)dt \tag{27}$$

Only the first series in Eq. (26) contributes to the singular part of the moment field. Substituting Eq. (27) into this series, the singular part may be written as

$$M_{rr}|_{r=a} = qa^2\phi(\alpha)/(2\varepsilon\alpha)^{1/2} \tag{28}$$

where $\varepsilon=\theta-\alpha$, $0<\varepsilon\ll 1$.

The bending stress normal to the circular boundary along the clamped segment is given by

$$\sigma_{rr} = 12M_{rr}\delta/h^3 \tag{29}$$

In the vicinity of the mixed condition

$$\sigma_{rr} = k/\varepsilon^{1/2} \tag{30}$$

where

$$k = 12\delta qa^2\phi(\alpha)/h^3(2\alpha)^{1/2} \tag{31}$$

is the bending stress intensity factor.

Simply Supported—Free Circular Plate

Considering one quadrant of the circular plate for solutions exhibiting symmetry about two axes, the boundary conditions may be expressed as follows on $r = a$:

$$M_{rr} = -D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] = 0, \quad 0 \leqq \theta \leqq \frac{\pi}{2} \tag{32}$$

$$\partial w/\partial\theta = 0, \qquad \alpha < \theta \leqq \pi/2 \tag{33}$$

$$V_r = -D\left\{\frac{\partial}{\partial r}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right) + \frac{1}{r}(1-\nu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial\theta^2}\right)\right\} = 0, \quad 0 \leqq \theta < \alpha \tag{34}$$

Boundary condition (32) leads to

$$B_0 = -(3+\nu)qa^2/(1+\nu)32D \tag{35}$$

and

$$A_m = h_m a^2 B_m, \qquad m = 2,4,\ldots \tag{36}$$

where

$$h_m = -[(1-\nu)m^2 + (3+\nu)m + 2(1+\nu)]/[(1-\nu)m(m-1)] \tag{37}$$

Application of boundary conditions (33) and (34) leads to

$$\sum_{m=2,4,\ldots}^{\infty} P_m \sin m\theta = 0, \qquad \alpha < \theta \leqq \frac{\pi}{2} \tag{38}$$

$$\sum_{m=2,4,\ldots}^{\infty} m^2\left[1 - \frac{(1+\nu)}{2m} + F_m\right] P_m \cos m\theta = \frac{1}{(3+\nu)(1-\nu)} \qquad 0 \leqq \theta < \alpha \tag{39}$$

where

$$B_m = a^{2-m} qP_m/[m(h_m+1)D] \tag{40}$$

and

$$F_m = [(\nu+3)(\nu-1)]/2m[2m+1+\nu] \tag{41}$$

The solution to these dual series equations proceeds by use of the representation

$$P_m = \int_0^\alpha t\phi(t)J_1(mt)\,dt \tag{42}$$

which automatically satisfies the first of the dual series equations by virtue of Eq. (5), and provides a square root moment singularity just outside of the support region. Although the mixed conditions lie on a circular boundary, the nature of the singularity must be and is in agreement with the standard references for related problems.[13–17]

The second of the dual series equations may be modified to appear as

$$-\frac{d^2}{d\theta^2}\sum_{m=2,4,\ldots}^{\infty} P_m \cos m\theta - \frac{1+\nu}{2}\frac{d}{d\theta}\sum_{m=2,4,\ldots}^{\infty} P_m \sin m\theta + \sum_{m=2,4,\ldots}^{\infty} m^2 F_m P_m \cos m\theta = \frac{1}{(3+\nu)(1-\nu)}, \qquad 0 \leqq \theta < \alpha \tag{43}$$

Substituting Eq. (42), integrating twice with respect to θ, and utilizing Eqs. (5) and (6) leads to

$$\int_0^\theta (\theta^2 - t^2)^{-1/2}\phi(t)\,dt = \frac{2}{\theta}\left\{\frac{1}{2}\int_0^\alpha \phi(t)\,dt + \frac{(1+\nu)}{4}\int_0^\theta s\int_s^\alpha (t^2-s^2)^{-1/2}\phi(t)\,dt\,ds - \int_0^\alpha t\phi(t)\int_0^\infty [\exp(\pi s)-1]^{-1} I_1(ts)\sinh(\theta s)\,ds\,dt + \int_0^\alpha t\phi(t)\sum_{m=2,4,\ldots}^{\infty} F_m J_1(mt)\cos(m\theta)\,dt + \frac{\theta^2}{2(3+\nu)(1-\nu)}\right\} \tag{44}$$

By use of Abel's integral equation and with the help of certain identities,[11,12] the final result may be written in the form of Eq. (20) where

$$K(\rho,r) = \frac{(1+\nu)\alpha}{\pi}\begin{cases}(r/\rho)[E(\rho/r) - F(\rho/r)], & r > \rho \\ [E(r/\rho) - F(r/\rho)], & r < \rho\end{cases} + 2\alpha^2 r\int_0^\infty [\exp(\pi s)-1]^{-1} I_1(\alpha rs)s\left[\frac{2}{\pi} + L_1(\alpha\rho s)\right]ds + 2\alpha^2 r\sum_{m=2,4,\ldots}^{\infty} F_m J_1(m\alpha r)mJ_1(m\alpha\rho) \tag{45}$$

and

$$f(\rho) = \alpha\rho/(3+\nu)(1-\nu) \tag{46}$$

In deriving the dual series equations for this case, the condition $\partial w/\partial\theta = 0$, Eq. (33), was used because it permitted the dual series equations to be cast into the proper form for solution. To satisfy the condition that $w = 0$ along the support region, the coefficient A_0 needs to be evaluated. This is accomplished by use of the condition $w = 0$ at $r = a$, $\theta = \pi/2$. Substitution of Eq. (7) into the resulting expression for w leads to

$$A_0 = \frac{qa^4}{D}\left[\frac{(5+\nu)}{64(1+\nu)} + \frac{1}{4}\alpha^3\int_0^1 \rho^2\psi(\rho)\,d\rho\right] \tag{47}$$

The deflection at the center of the plate is equal to A_0; the bending moments at the center are given by

$$M_{rr}\Big|_{\substack{r=0\\ \theta=0}} = qa^2\left[\frac{3+\nu}{16} + \frac{(\nu-1)}{h_2+1}h_2 P_2\right] \tag{48}$$

and

$$M_{\theta\theta}\Big|_{\substack{r=0\\ \theta=0}} = qa^2\left[\frac{3+\nu}{16} + \frac{(1-\nu)}{h_2+1}h_2 P_2\right] \tag{49}$$

Numerical Results

In order to evaluate the Fourier series coefficients for the physical quantities of interest, a numerical solution of Eq. (20) must be obtained. Since the kernel of Eq. (20) is infinite for $r = \rho$, a special procedure[18] is used to replace the integral equation with a finite number of algebraic equations. Writing Eq. (20) in the form

$$\psi(\rho) + \int_0^1 K(\rho,r)[\psi(r) - \psi(\rho)]\,dr + \psi(\rho)\int_0^1 K(\rho,r)\,dr = f(\rho), \qquad 0 \leqq \rho \leqq 1 \tag{50}$$

it is seen that the logarithmic singularity at $r = \rho$ in the first integral is eliminated due to the factor $[\psi(r) - \psi(\rho)]$. The second integral in Eq. (50), illustrated here for the clamped—simply supported case, can be written as

$$\int_0^1 K(\rho,r)\,dr = \int_0^1\left\{K(\rho,r) + \begin{matrix}\left(\dfrac{1+\nu}{\pi}\right)\dfrac{\alpha\rho}{r}\log\left[1-\left(\dfrac{\rho}{r}\right)^2\right]^{1/2}, & r > \rho \\ \left(\dfrac{1+\nu}{\pi}\right)\alpha\log\left[1-\left(\dfrac{r}{\rho}\right)^2\right]^{1/2}, & r < \rho\end{matrix}\right\}dr - \left(\frac{1+\nu}{\pi}\right)\alpha\int_0^\rho \log\left[1-\left(\frac{r}{\rho}\right)^2\right]^{1/2} dr - \left(\frac{1+\nu}{\pi}\right)\alpha\int_\rho^1\left(\frac{\rho}{r}\right)\log\left[1-\left(\frac{\rho}{r}\right)^2\right]^{1/2} dr \tag{51}$$

Table 1 Physical quantities for clamped—simply supported case

$\alpha°$	wD/qa^4	wD/qa^{4a}	wD/qa^{4b}	M_{rr}/qa^2	$M_{\theta\theta}/qa^2$	M_{rr}/qa^{2b}	$M_{\theta\theta}/qa^{2b}$	$\psi(1)$
0	0.0156	0.0156	0.0156	0.0812	0.0812	0.0812	0.0812	0
11.25	0.0162	0.0162	...	0.0812	0.0842	...	...	−0.0229
15.00	0.0166	...	0.0166	0.0812	0.0863	0.0810	0.0864	−0.0299
22.50	0.0176	0.0178	...	0.0813	0.0917	...	...	−0.0434
30.00	0.0191	...	0.0192	0.0818	0.0986	0.0816	0.0995	−0.0563
33.75	0.0199	0.0200	...	0.0824	0.1024	...	...	−0.0627
45.00	0.0229	0.0230	0.0235	0.0856	0.1147	0.0858	0.1175	−0.0819
56.25	0.0266	0.0268	...	0.0921	0.1277	...	...	−0.1021
60.00	0.0281	...	0.0288	0.0953	0.1321	0.0958	0.1351	−0.1095
67.50	0.0313	0.0316	...	0.1033	0.1409	...	...	−0.1262
75.00	0.0353	...	0.0366	0.1145	0.1502	0.1177	0.1534	−0.1479
78.75	0.0376	0.0381	...	0.1217	0.1552	...	...	−0.1630
89.00	0.0509	0.0530	...	0.1648	0.1809	...	...	−0.4439

[a] From Ref. 2.
[b] From Ref. 3.

The integrand of the first term on the right-hand side of Eq. (51) is now regular everywhere. By use of an identity[11]

$$\int_0^\rho \log\left[1-\left(\frac{\rho}{r}\right)^2\right]^{1/2} dr = \rho(\log 2 - 1) \qquad (52)$$

A closed form expression for the last integral in Eq. (51) was not found. However, by a change of variables, it is convenient to write

$$\int_\rho^1 \left(\frac{\rho}{r}\right)\log\left[1-\left(\frac{\rho}{r}\right)^2\right]^{1/2} dr = \rho\int_0^{(1-\rho^2)^{\frac{1}{2}}} \frac{u\log u\, du}{(1-u^2)}, \qquad \rho < 1 \qquad (53)$$

A numerical integration of the right-hand side of Eq. (53) is easily performed, since the singularity at $r = \rho$ has been removed.

By use of Simpson's rule in conjunction with the above procedure, the integral equation was solved for $\psi(\rho)$. Eleven simultaneous equations were used to obtain the results shown in the Tables 1 and 2. As few as five equations were found to give accurate results in many cases.

Table 1 shows the deflections and bending moments at the center of a uniformly loaded plate for the clamped—simply supported case. The tabulated results are seen to be in excellent agreement with the numerical treatments of Conway and Farnham[2] and Leissa and Clausen.[3] In the vicinity of the mixed condition, however, such excellent agreement would not be expected.[19,20] It should be remarked that the present method provides results which are in accord with classical plate theory. In order to accurately describe the behavior of the physical quantities at the point of discontinuity, a higher order theory would have to be utilized.

Table 2 shows the results at the center of the plate for the simply supported—free case. A comparison of the tabulated deflections with Fig. 6 of Conway and Farnham[2] reveals excellent agreement except when the support segments are very short. From Eq. (g) on page 294 of Timoshenko and Woinowsky-Krieger,[6] the upper bound, which corresponds to a circular plate point supported on opposite sides of a diameter, is computed to be

$$w = 0.2809\, qa^4/D$$

In addition to the deflection and bending moments, the auxiliary function $\psi(1)$ is also tabulated in Tables 1 and 2. For the clamped—simply supported case, Eq. (31) can be used to obtain the stress intensity factor. As the clamped segment becomes very short, it is seen that the stress intensity factor increases rapidly. An expression for the stress intensity factor similar to Eq. (31), but proportional to $\alpha^{1/2}\phi(\alpha)$, can be developed for the simply supported—free case. Table 2 indicates that the stress intensity factor for this case increases and reaches a maximum when $\alpha \approx 80°$, i.e., when the angle subtended by the support segments is approximately 20°.

References

[1] Nowacki, W. and Olesiak, Z., "Vibration, Buckling and Bending of a Circular Plate Clamped along Part of Its Periphery and Simply Supported on the Remaining Part," *Bulletin de l'Academie Polonaise des Sciences,* Cl. IV. *Applied Mechanics,* Vol. 4, No. 4, 1956, pp. 247–258.

[2] Conway, H. D. and Farnham, K. A., "Deflections of Uniformly Loaded Circular Plates with Combinations of Clamped, Simply Supported and Free Boundary Conditions," *International Journal of Mechanical Sciences,* Vol. 9, 1967, pp. 661–671.

[3] Leissa, A. W. and Clausen, W. E., "Deflection of a Circular Plate Having Mixed Boundary Conditions," *AIAA Journal,* Vol. 5, No. 12, Dec. 1967, pp. 2287–2288.

[4] Bartlett, C. C., "The Vibration and Buckling of a Circular Plate Clamped on Part of Its Boundary and Simply Supported on the Remainder," *Quarterly Journal of Mechanics and Applied Mathematics,* Vol. 16, 1963, pp. 431–440.

[5] Noble, B., "The Vibration and Buckling of a Circular Plate Clamped on Part of Its Boundary and Simply Supported on the Remainder," *Developments in Mechanics,* Vol. 3, Pt. 2: *Dynamics and Fluid Mechanics, Proceedings of the Ninth Midwestern Mechanics Con-*

Table 2 Physical quantities for simply supported—free case

$\alpha°$	wD/qa^4	M_{rr}/qa^2	$M_{\theta\theta}/qa^2$	$\psi(1)$	$(\alpha)^{1/2}$ rad	$(\alpha)^{1/2}\psi(1)$
0	0.0637	0.2063	0.2063	0	0	0
15	0.0638	0.2058	0.2067	0.1182	0.5117	0.0605
30	0.0661	0.1995	0.2130	0.2489	0.7236	0.1801
45	0.0767	0.1728	0.2397	0.3956	0.8862	0.3506
60	0.1086	0.1082	0.3043	0.5557	1.023	0.5685
70	0.1508	0.0413	0.3712	0.6565	1.105	0.7254
80	0.2130	−0.0336	0.4461	0.7075	1.182	0.8363
85	0.2471	−0.0642	0.4767	0.6604	1.218	0.8044
89	0.2678	−0.0783	0.4908	0.4792	1.246	0.5971

ference, Univ. of Wisconsin, Madison, Wis., April 16–18, 1965, Wiley, New York, 1967, pp. 141–146.

[6] Timoshenko, S. and Woinowsky-Krieger, S., *Theory of Plates and Shells*, 2nd ed., McGraw-Hill, New York, 1959.

[7] Watson, G. N., *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, England, 1962, p. 405.

[8] Buchwald, V. T., "A Mixed Boundary-Value Problem in the Elementary Theory of Elastic Plates," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 10, 1957, pp. 183–190.

[9] Gladwell, G. M. L., "Some Mixed Boundary Value Problems in Isotropic Thin Plate Theory," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 11, 1958, pp. 159–171.

[10] Werner, H. and Peter, J., "Beitrag zur Berechnung von Platten mit teilweiser Randeinspannung und teilweise einspannungsfreier Auflagerung," *Der Stahlbau*, Vol. 35, No. 4, 1966, pp. 97–106.

[11] Gradshteyn, I. S. and Ryzhic, I. M., *Table of Integrals, Series and Products*, 4th ed., Academic Press, New York, 1965, pp. 427, 584.

[12] Dwight, H. B., *Tables of Integrals and Other Mathematical Data*, 4th ed., Macmillan, New York, 1969, pp. 179–186.

[13] Goodier, J. N., "The Influence of Circular and Elliptical Holes on the Transverse Flexure of Elastic Plates," *Philosophical Magazine*, Vol. 22, 1936, pp. 69–80.

[14] Sih, G. C., Paris, P. C., and Erdogan, F., "Crack-Tip, Stress-Intensity Factors for Plane Extension and Plate Bending Problems," *Journal of Applied Mechanics*, Vol. 29, 1962, pp. 306–312.

[15] Yu, Y. Y., "The Influence of a Small Hole or Rigid Inclusion on the Transverse Flexure of Thin Plates," *Proceedings of the Second U.S. National Congress of Applied Mechanics*, 1954, pp. 381–387.

[16] Williams, M. L., "The Bending Stress Distribution at the Base of a Stationary Crack," *Journal of Applied Mechanics*, Vol. 28, 1961, pp. 78–82.

[17] Williams, M. L., "Surface Stress Singularities Resulting from Various Boundary Conditions in Angular Corners of Plates Under Bending," *U.S. National Congress of Applied Mechanics*, Illinois Inst. of Technology, Chicago, Ill., June 1952, pp. 325–329.

[18] Kantorovich, L. V. and Krylov, V. I., *Approximate Methods of Higher Analysis*, Interscience, New York, 1958, pp. 101–102.

[19] Leissa, A. W., Clausen, W. E., Hulbert, L. E., and Hopper, A. T., "A Comparison of Approximate Methods for the Solution of Plate Bending Problems," *AIAA Journal*, Vol. 7, No. 5, May 1969, pp. 920–928.

[20] Stahl, B. and Keer, L. M., "Vibration and Stability of Cracked Rectangular Plates," *International Journal of Solids and Structures*, Vol. 8, 1972, pp. 69–91.

JUNE 1972 AIAA JOURNAL VOL. 10, NO. 6

Modeling of the Turbulence Structure of the Atmospheric Surface Layer

T. R. SUNDARAM,* G. R. LUDWIG,† AND G. T. SKINNER‡
Cornell Aeronautical Laboratory Inc., Buffalo, N.Y.

Theoretical and experimental concepts relevant to the laboratory simulation of the turbulence characteristics of the atmospheric surface layer are discussed. It is argued that, for proper simulation, the laboratory flow must satisfy the requirements of horizontal homogeneity, aerodynamically rough flow and proper thermal stratification. A discussion is also given of the manner in which the similitude criteria can be used to derive the dimensions of a laboratory facility required for proper modelling. Experiments on the generation of laboratory flows which can be used to simulate neutrally-stable atmospheric surface layers are described. A fairly simple technique, in which the flow near the floor of a wind tunnel is "tripped" by a flat-plate fence spanning the width of the tunnel and then allowed to develop over a rough ground, is shown to provide suitable flows provided that the height of the fence is carefully matched to the aerodynamic roughness of the roughness elements. The turbulence characteristics of the laboratory flows generated using the present technique compare very well with the corresponding characteristics of atmospheric-surface-layer flows.

Introduction

IN recent years increasing interest has developed in various environmental problems which are influenced, as well as controlled, by the boundary-layerlike behavior of the atmospheric motions near the Earth's surface. Many of the natural phenomena that are crucial to the support of life on the Earth take place within the planetary boundary layer. This layer is a highly complex entity, and the motions within it are generally turbulent and notoriously difficult to analyze. It is impossible to obtain satisfactory theoretical solutions to certain problems, such as the diffusion of emissions from a low stack on a building in the lee of a large complex of buildings. These problems are influenced not only by the atmospheric turbulence field, but also by the turbulence generated by the buildings themselves. Although actual observations in the atmosphere have yielded some results on such problems, much insight into the relevant phenomena has not been gained since conditions in the atmosphere are essentially uncontrolled and uncontrollable. Thus, laboratory modeling of the turbulence structure of the lower atmosphere in a wind tunnel like facility seems to hold much promise.

The use of wind tunnels to study atmospheric problems has a fairly long history. The early basic studies on turbulent boundary layers in wind tunnels also shed much light on similar problems in the atmosphere. For example, the well-known logarithmic law for the wind profile in a neutrally stable atmosphere was deduced from Nikuradse's experiments on the flow in rough-walled pipes and conduits.[1,2] A number of early attempts were also made to use conventional, aeronautical-type wind tunnels to model

Presented as Paper 71-136 at the AIAA 9th Aerospace Sciences Meeting, New York, January 25–27, 1971; submitted March 24, 1971; revision received February 15, 1972. This paper is based upon research sponsored by the U.S. Atomic Energy Commission Contract AT (30-1)-4038 and by CAL Internal Research (W/A 86-232).

Index category: Atmospheric, Space, and Oceanographic Sciences.

* Principal Aeronautical Engineer; present address Hydronautics Inc., Laurel, Md. Member AIAA.

† Principal Aeronautical Engineer. Member AIAA.

‡ Principal Aeronautical Engineer.